Meromorphic continuation and non-polar singularities of local zeta functions in some smooth cases

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Introduction

Introduction

We consider the following integrals

$$Z_f(\varphi)(s) := \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) dx \quad \text{for } s \in \mathbb{C}$$

where

- f is a real-valued (C^∞) smooth function defined on an open neighborhood U of the origin in $\mathbb{R}^n$,
- φ is a real-valued (C^∞) smooth function defined on $\mathbb{R}^n$ and the support of φ is contained in U.

The integrals $Z_f(\varphi)(s)$ converge locally uniformly on $\mathrm{Re}(s) > 0$.
$\Rightarrow Z_f(\varphi)(s)$ become holomorphic functions on $\mathrm{Re}(s) > 0$,
which are called *local zeta functions*.

We assume that

$$f(0) = 0 \text{ and } \nabla f(0) = 0.$$

We also assume that the support of φ is sufficiently small.

In the case when f is real analytic, there have been many precise investigations with respect to meromorphic continuation of local zeta functions.

Example (I.M.Gel'fand-Shilov (1964, published in Russian in 1958))

Let $n = 1$ and $f(x) = x^a$, where $a \in \mathbb{N}$. Then $Z_f(\varphi)(s)$ can be meromorphically continued to $\mathbb{C}$, and its poles are contained in $\{-j/a : j \in \mathbb{N}\}$.

Proof) Let $r > 0$ satisfying that the interval $(-r, r)$ containing the support of φ. We decompose $Z_f(\varphi)(s)$ as follows:

$$
\begin{aligned}
Z_f(\varphi)(s) &= \int_{-r}^{r} |x^a|^s \varphi(x) dx \\
&= \int_0^r |x^a|^s \varphi(x) dx + \int_{-r}^{0} |x^a|^s \varphi(x) dx \\
&= \int_0^r x^{as} \varphi(x) dx + \int_0^r x^{as} \varphi(-x) dx.
\end{aligned}
$$

Here, Taylor's formula gives

$$\varphi(\pm x) = \sum_{j=0}^{N} c_j x^j + x^{N+1} R(x),$$

where c_j are the Taylor' coefficients of $\varphi(\pm x)$ and $R(x)$ is a smooth function. Hence, we have

$$\begin{aligned}\int_0^r x^{as}\varphi(\pm x)dx &= \sum_{j=0}^{N} c_j \int_0^r x^{as+j}dx + \int_0^r x^{as+N+1}R(x)dx \\ &= \sum_{j=0}^{N} \frac{c_j r^{as+j+1}}{as+j+1} + \int_0^r x^{as+N+1}R(x)dx.\end{aligned}$$

The last integral becomes a holomorphic function on $\mathrm{Re}(s) > -(N+1)/a$. Taking $N \to \infty$, we have the assertion.

Conjecture (I.M.Gel'fand (1954 ICM in Amsterdam))

Let f be a polynomial. Then $Z_f(\varphi)(s)$ can be meromorphically continued to $\mathbb{C}$.

Theorem (Bernstein-S.I.Gel'fand (1969), Atiyah (1970))

Let f be real analytic. Then $Z_f(\varphi)(s)$ can be meromorphically continued to $\mathbb{C}$, and its poles are contained in finitely many arithmetic progressions which consist of negative rational numbers.

The theorem is proven by using Hironaka's resolution of singularities. Recently, Greenblatt (2010) obtains the same result by constructing elementary resolution of singularities.

Varchenko (1976) gives the exact location of poles of local zeta functions by using the theory of toric varieties based on the *Newton polyhedron* of f under some nondegeneracy condition.

Definition

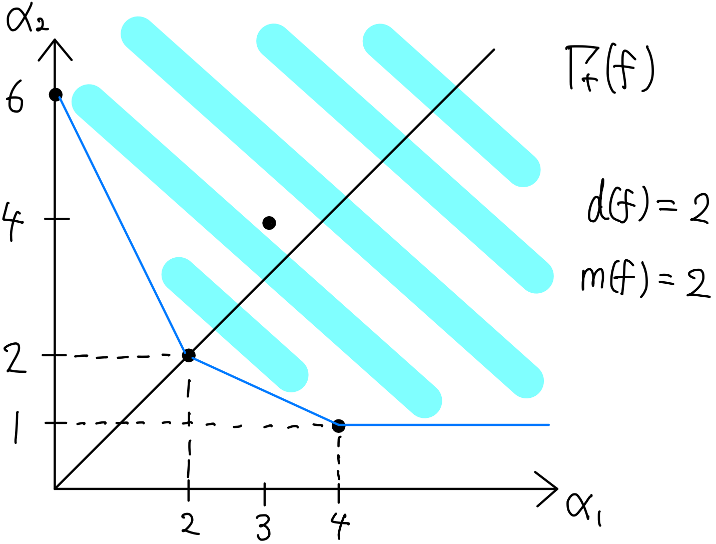
Let f be a real-valued smooth function near the origin and $\sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} c_\alpha x^\alpha$ the Taylor series of f. The Newton polyhedron $\Gamma_+(f)$ of f is defined to be the convex hull of $\cup_{\alpha \in S_f}(\alpha + (\mathbb{R}_{\geq 0})^n)$, where $S_f = \{\alpha \in (\mathbb{Z}_{\geq 0})^n : c_\alpha \neq 0\}$.

Definition

Assume $\Gamma_+(f) \neq \emptyset$. Let α_0 be the point at which the line $\alpha_1 = \cdots = \alpha_n$ in $\mathbb{R}^n$ intersects the topological boundary of $\Gamma_+(f)$. The coordinate of α_0 is called the Newton distance of f, which is denoted by $d(f)$. The codimension of the face of $\Gamma_+(f)$ whose relative interior contains α_0 is called the Newton multiplicity of f, which is denoted by $m(f)$.

Newton polyhedra

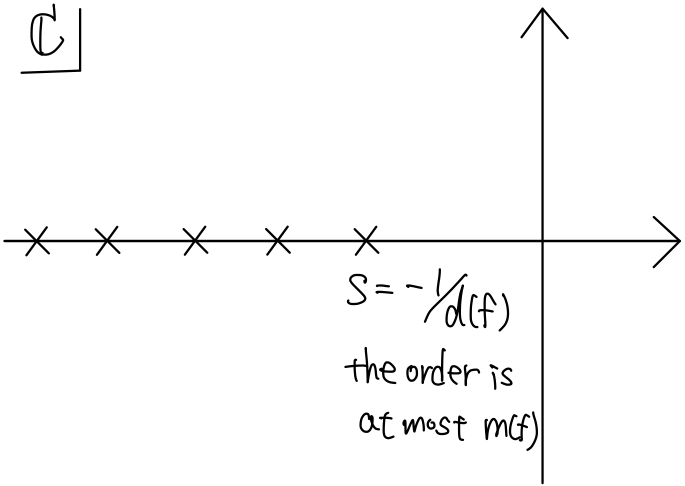
$$f(x,y) = x^2y^2 + x^4y + x^3y^4 + y^6$$



Theorem (Varchenko (1976))

Assume that f is real analytic and "nondegenerate with respect to its Newton polyhedron". Then the poles of $Z_f(\varphi)(s)$ are contained in the subset $S \subset \mathbb{Q}_{<0}$ which are obtained by using the theory of toric varieties based on the geometry of the Newton polyhedron of f. In particular, if $d(f) > 1$, then $\max S = -1/d(f)$. Moreover, the order of the pole at $s = -1/d(f)$ is equal to at most $m(f)$.

Varchenko's results



There have been many precise investigations in the real analytic case.

- Denef and Sargos (1989, 1992), Denef, Laeremans and Sargos (1997), Denef, Nicaise and Sargos (2005)
  $\cdots$ improvement of Varchenko's results. coefficients of the first pole
- Okada and Takeuchi (2013)
  $\cdots$ calculate the coefficients of poles
- Collins, Greenleaf and Pramanik (2013)
  $\cdots$ resolution of singularities by explicit and elementary approach

- Kamimoto and N. (2016) generalize Varchenoko's results when f belongs to a certain class of smooth functions.

Smooth cases

Let f be smooth. Holomorphic continuation of $Z_f(\varphi)(s)$ relates to the famous index

$$c_0(f) := \sup\left\{\mu > 0 : \begin{array}{l}\text{there exists an open neighborhood } V \text{ of} \\ \text{the origin in } U \text{ such that } |f|^{-\mu} \in L^1(V)\end{array}\right\},$$

which is called *log canonical threshold* or *critical integrability index*.

$Z_f(\varphi)(s)$ converges locally uniformly on $\mathrm{Re}(s) > -c_0(f)$
$\Rightarrow Z_f(\varphi)(s)$ becomes a holomorphic function on $\mathrm{Re}(s) > -c_0(f)$.

Theorem (Kamimoto-N. (2019))

If $\varphi(0) > 0$ and $\varphi(x) \geq 0$ on U, then $Z_f(\varphi)(s)$ cannot be holomorphically continued to any open neighborhood of $s = -c_0(f)$.

In other words, $Z_f(\varphi)(s)$ has a singularity at $s = -c_0(f)$.

In Varchenko's results, we have the equality: $c_0(f) = 1/d(f)$.

Theorem (Greenblatt (2006))

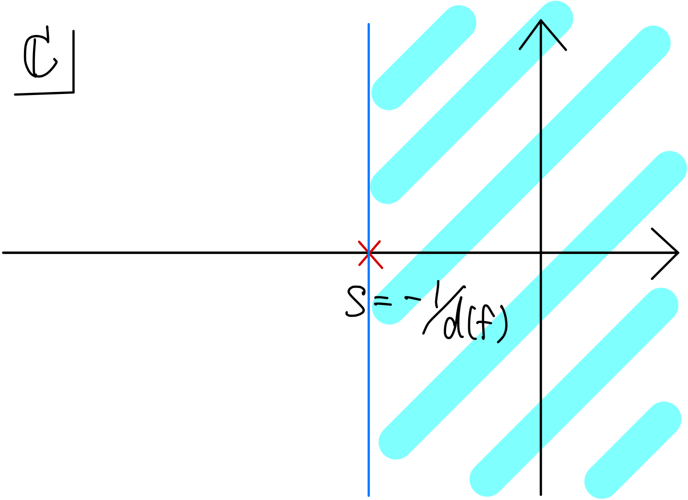
When f is a nonflat smooth function defined near the origin in $\mathbb{R}^2$, the equation $c_0(f) = 1/d(f)$ holds in "adapted coordinates".

Ikromov and Müller (2011) give equivalent conditions for adaptedness of smooth coordinates in dimension two.

Remark

In 2-dimensional adapted coordinates, $Z_f(\varphi)(s)$ becomes a holomorphic function on $\mathrm{Re}(s) > -1/d(f)$ and has a singularity at $s = -1/d(f)$ under the condition $\varphi(0) > 0$, $\varphi(x) \geq 0$ on U.

Holomorphic continuation of $Z_f(\varphi)(s)$



In this talk, we consider the following quantities

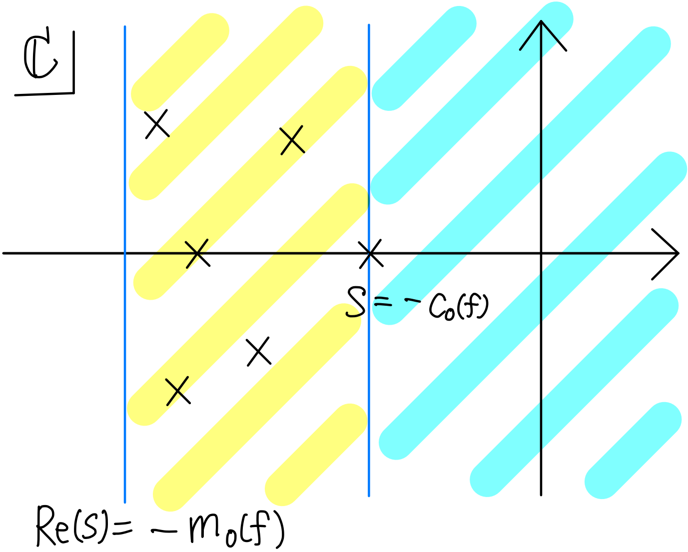
$$m_0(f,\varphi) := \sup \left\{ \rho > 0 : \begin{array}{l} \text{The domain in which } Z_f(\varphi) \text{ can} \\ \text{be meromorphically continued} \\ \text{contains the half-plane } \mathrm{Re}(s) > -\rho \end{array} \right\},$$
$$m_0(f) := \inf \left\{ m_0(f,\varphi) : \varphi \in C_0^\infty(U) \right\}.$$

$Z_f(\varphi)(s)$ can be meromorphically continued to $\mathrm{Re}(s) > -m_0(f)$ for any $\varphi \in C_0^\infty(U)$.

We have $c_0(f) \leq m_0(f)$.

f is real analytic $\Rightarrow m_0(f) = \infty$.

The quantity $m_0(f)$



Theorem (Kamimoto-N. (2019))

Let

$$f(x,y) = x^a y^b + x^a y^{b-q} e^{-1/|x|^p}, \tag{1}$$

where a, b, p, q *satisfy that*

- a, b *are nonnegative integers satisfying* $a < b, 2 \leq b$,
- p *is a positive real number,*
- q *is an even integer satisfying* $2 \leq q \leq b$.

Denote $\sigma := \mathrm{Re}(s)$. *Then the following hold:*

(i) *If* $p > 1 - a/b$, *then*

$$\lim_{\sigma \to -1/b+0} (b\sigma + 1)^{1 - \frac{1-a/b}{p}} \cdot Z_f(\varphi)(\sigma) = 4B \cdot \varphi(0,0),$$

where B *is the positive constant defined by* $B = \int_0^\infty x^{-a/b} \left(1 - e^{-1/(qx^p)}\right) dx$. *Note that the above improper integral converges.*

(ii) If $p = 1 - a/b$, then

$$\lim_{\sigma \to -1/b+0} |\log(b\sigma + 1)|^{-1} \cdot Z_f(\varphi)(\sigma) = \frac{4}{pq} \cdot \varphi(0,0).$$

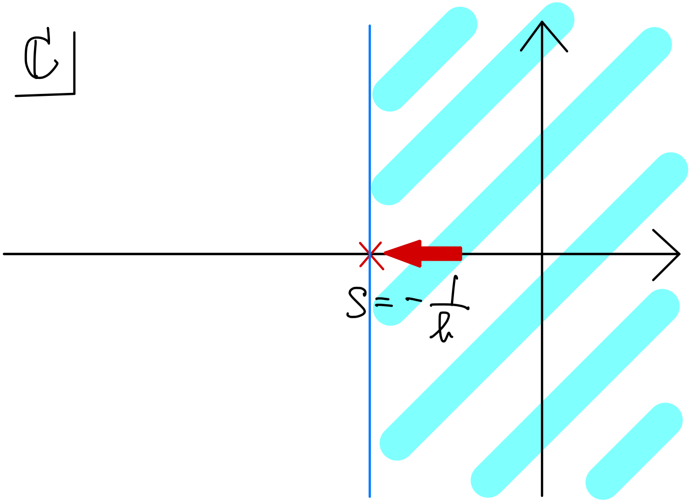
(iii) If $0 < p < 1 - a/b$, then there exists a constant $B(\varphi)$ which depends on f, φ but is independent of σ such that

$$\lim_{\sigma \to -1/b+0} Z_f(\varphi)(\sigma) = B(\varphi).$$

Here $B(\varphi)$ is positive if $\varphi(0,0) > 0$ and $\varphi(x,y) \geq 0$.

The theorem implies if f be as in (1), then $m_0(f) = 1/b$.

Non-polar singularities of $Z_f(\varphi)(s)$



A classification of smooth model cases

A classification of smooth model cases

We consider the smooth functions f of the form

$$f(x, y) = u(x, y)x^a y^b + (\text{flat function at the origin}), \tag{2}$$

where $a, b \in \mathbb{Z}_{\geq 0}$ satisfying $a \leq b$ and u is a smooth function defined near the origin with $u(0, 0) \neq 0$, which are regarded as a model of smooth functions in dimension two.

Definition

We say a smooth function defined near the origin is flat at the origin if all the derivatives of the function vanish at the origin.

In the smooth model cases, $c_0(f) = 1/d(f) = 1/b$ and $m_0(f) \geq 1/b$.

Lemma (Classification of smooth model cases)

Let f be as in (2). If U is sufficiently small, then f can be expressed on U as one of the following four forms.

(A) $f(x,y) = v(x,y)x^a y^b$,

(B) $f(x,y) = v(x,y)x^a y^b + g(x,y)$,

(C) $f(x,y) = v(x,y)x^a y^b + h(x,y)$,

(D) $f(x,y) = v(x,y)x^a y^b + g(x,y) + h(x,y)$,

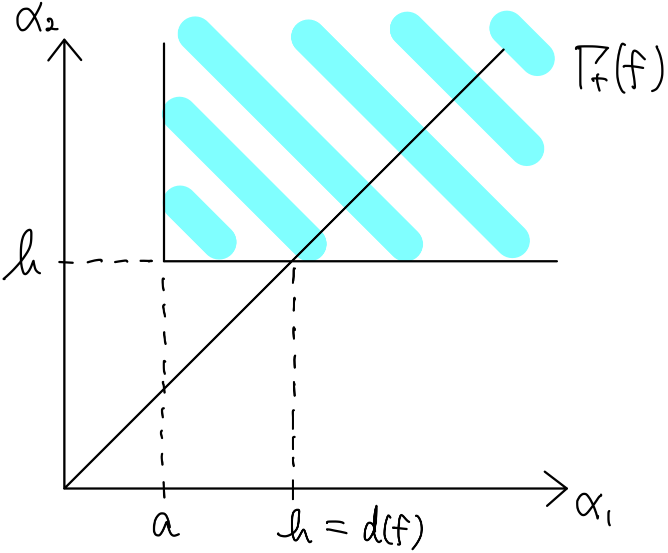
where $a \leq b$ and v, g, h are smooth functions defined on U satisfying the following properties:

- $v(0,0) = u(0,0) \neq 0$.
- *g, h are non-zero flat functions admitting the forms:*

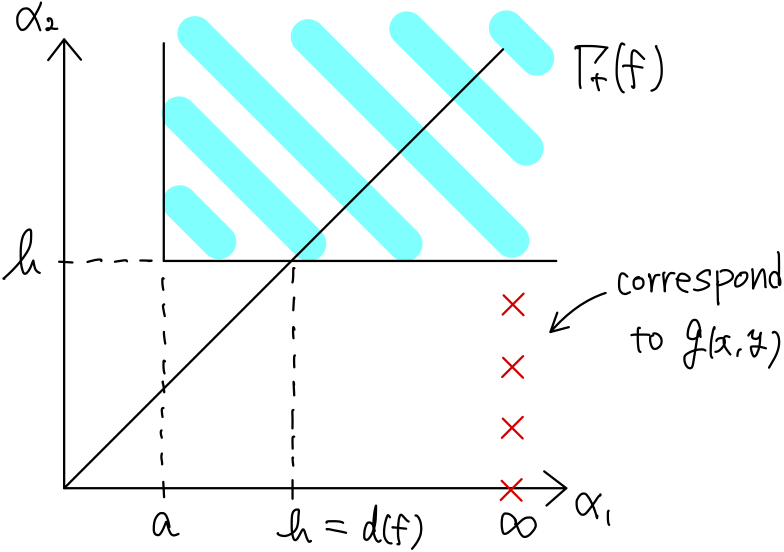
$$g(x,y) = \sum_{j=0}^{b-1} y^j g_j(x) \quad \text{and} \quad h(x,y) = \sum_{j=0}^{a-1} x^j h_j(y),$$

where g_j, h_j are flat at the origin.

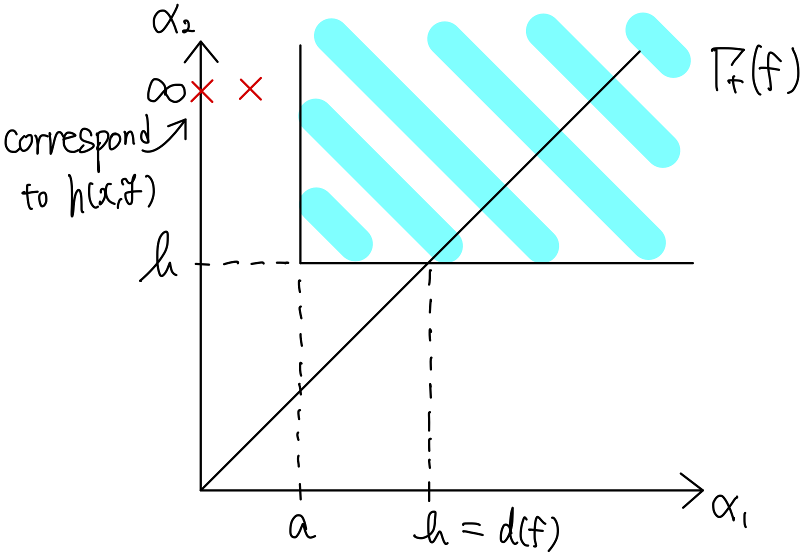
Newton polyhedra of smooth models in the case (A)



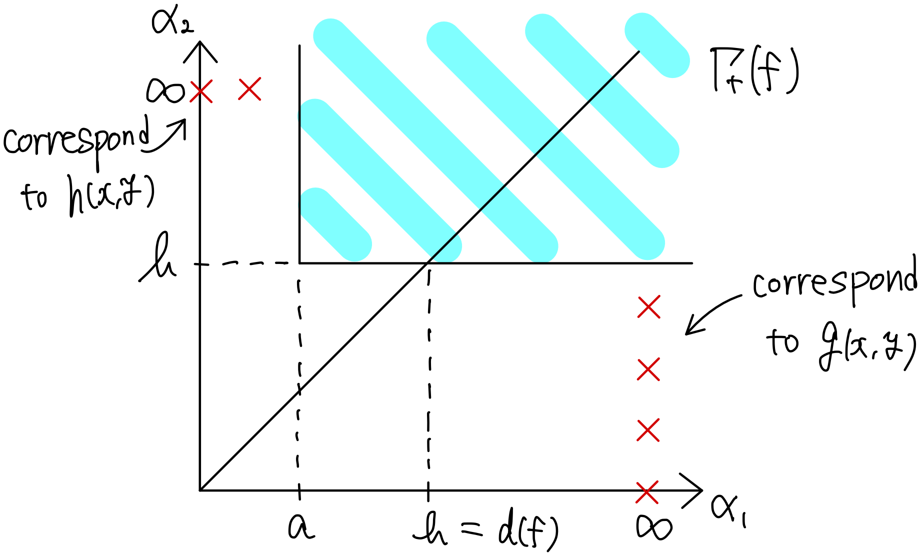
Newton polyhedra of smooth models in the case (B)



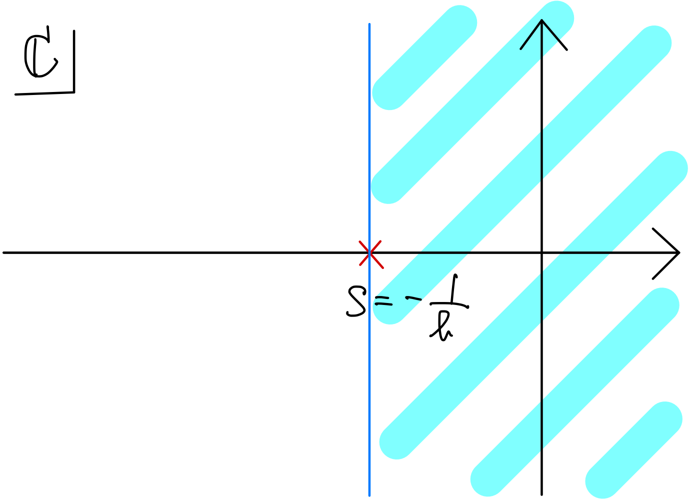
Newton polyhedra of smooth models in the case (C)



Newton polyhedra of smooth models in the case (D)



Holomorphic continuation in smooth model cases



Remark

The cases (A), (B), (C), (D) are related as follows.

1. Under the condition: $b = 0$, which implies $a = 0$, the cases (B), (C), (D) belong to the case (A).
2. The case (B) with the condition: $b = 1 \Rightarrow$ The case (A)
3. The case (C) with the condition: $a = 0, 1 \Rightarrow$ The case (A).
4. The case (D) with the condition: $a = 0, 1 \Rightarrow$ The case (B).
5. The case (D) with the condition: $b = 1 \Rightarrow$ The case (A).

When f belongs to the case (A), f can be expressed as a monomial under a smooth coordinate change on $\mathbb{R}^2$, which gives that $m_0(f) = \infty$ holds.

In the case (C), we have a nontrivial lower estimate of $m_0(f)$ as follows.

Theorem (Kamimoto-N. (2020))

Let f be as in (C). Then $Z_f(\varphi)(s)$ can be meromorphically continued to $\mathrm{Re}(s) > -1/a$, and its poles are contained in $\{-j/b : j \in \mathbb{N}$ with $j < b/a\}$. In particular, $m_0(f) \geq 1/a$ holds.

Meromorphic continuation in the case (C)

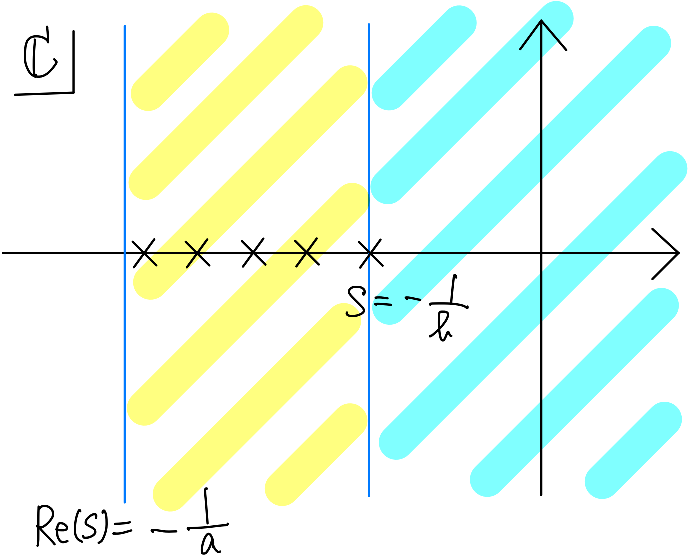


Table: The values of $m_0(f)$.

	(A)	(B)	(C)	(D)
$m_0(f)$	∞	$\geq 1/b$	$\geq 1/a$	$\geq 1/b$

Recently, Kamimoto introduces a certain quantity for a non-flat smooth function f and obtains a general lower estimate of $m_0(f)$ by using the quantity. His result implies the lower estimates in the table.

Question

Are the lower estimates of $m_0(f)$ in the above table optimal ?

Here, their optimalities mean that there exists a smooth function f of the each form such that the value of $m_0(f)$ attains the lower bound of $m_0(f)$.

In the case (B) with the condition: $a < b$, the optimality has been already shown.

Main results

Main results

In the case when $a < b$, we have the following.

Theorem 1 (N. arXiv:2206.10246)

(i) If f belongs to the case (C) with the condition: $a < b$, then the estimate: $m_0(f) \geq 1/a$ is optimal.

(ii) If f belongs to the case (D) with the condition: $a < b$, then the estimate: $m_0(f) \geq 1/b$ is optimal.

Key to proof of Theorem 1 (i)

We consider the following smooth functions belonging to the case (C):

$$f(x,y) = x^a y^b + x^{a-q} y^b e^{-1/|y|^p}, \tag{3}$$

where a, b, p, q satisfy that

- a, b are positive integers satisfying $2 \leq a \leq b$,
- p is a positive real number,
- q is an even integer satisfying $2 \leq q \leq a$.

We investigate the asymptotic behavior of $Z_f(\varphi)(s)$ as $\sigma := \mathrm{Re}(s) \to -1/a + 0$.

Theorem

Let f be as in (3). Then $Z_f(\varphi)(s)$ can be meromorphically continued to $\mathrm{Re}(s) > -1/a$, and its poles are contained in the set $\{-j/b : j \in \mathbb{N}$ with $j < b/a\}$. Moreover the following hold:

(i) *If one of the following hold:*
   (a) $0 < p < b/a - 1$,
   (b) b/a *is not an odd integer,*

   then

$$\lim_{\sigma \to -1/a+0} (a\sigma + 1)^{1+\frac{b/a-1}{p}} \cdot Z_f(\varphi)(\sigma) = \frac{-4pC}{q(b/a-1)} \cdot \varphi(0,0),$$

   where C is the positive constant defined by $C = \int_0^\infty u^{-b/a-p} e^{-1/(qu^p)} du$. Note that the above improper integral converges.

(ii) If $p = b/a - 1$ and b/a is an odd integer, then

$$\lim_{\sigma \to -1/a+0} (a\sigma + 1)^2 \cdot Z_f(\varphi)(\sigma) = \frac{-4C}{q} \cdot \varphi(0,0) + \frac{4a}{bp!} \cdot \frac{\partial^p \varphi}{\partial y^p}(0,0),$$

where C is as above.

(iii) If $p > b/a - 1$ and b/a is an odd integer, then

$$\lim_{\sigma \to -1/a+0} (a\sigma + 1)^2 \cdot Z_f(\varphi)(\sigma) = \frac{4a}{bm!} \cdot \frac{\partial^m \varphi}{\partial y^m}(0,0),$$

where $m = b/a - 1$.

Corollary

Let f be as in (3)*. Assume that one of the following hold:*

(a) $0 < p < b/a - 1$,

(b) b/a *is not an odd integer.*

If $(b/a - 1)/p$ is not an integer and $\varphi(0,0) \neq 0$, then $Z_f(\varphi)(s)$ cannot be meromorphically continued to any open neighborhood of $s = -1/a$ in $\mathbb{C}$. In particular, $m_0(f) = 1/a$ holds.

Remark

When $a = b$ in the above theorem, $Z_f(\varphi)(s)$ can be holomorphically continued to $\mathrm{Re}(s) > -1/a$ and

$$\lim_{\sigma \to -1/a+0} (a\sigma + 1)^2 \cdot Z_f(\varphi)(\sigma) = 4\varphi(0,0),$$

which does not answer whether $Z_f(\varphi)(s)$ has a singularity different from poles at $s = -1/a$.

Remark

The above corollary gives the optimality: $m_0(f) \geq 1/a$ in the case (C) with the condition: $2 \leq a < b$.

Key to proof of Theorem 1 (ii)

We also consider the following smooth functions belonging to the case (D):

$$f(x,y) = x^a y^b + x^a y^{b-q} e^{-1/|x|^p} + x^{a-\tilde{q}} y^b e^{-1/|y|^{\tilde{p}}}, \tag{4}$$

where $a, b, p, \tilde{p}, q, \tilde{q}$ satisfy that

- a, b are positive integers satisfying $2 \leq a < b$,
- p, $\tilde{p}$ are positive real numbers,
- q, $\tilde{q}$ are even integers satisfying $2 \leq q \leq b$, $2 \leq \tilde{q} \leq a$.

Theorem

Let f be as in (4). Then the following hold:

(i) *If $p > 1 - a/b$, then*

$$\lim_{\sigma \to -1/b+0} (b\sigma + 1)^{1-\frac{1-a/b}{p}} \cdot Z_f(\varphi)(\sigma) = 4\hat{C} \cdot \varphi(0,0),$$

where $\hat{C}$ is the positive constant defined by $\hat{C} = \int_0^\infty x^{-a/b} \left(1 - e^{-1/(qx^p)}\right) dx$. Note that the above improper integral converges.

(ii) *If $p = 1 - a/b$, then*

$$\lim_{\sigma \to -1/b+0} |\log(b\sigma + 1)|^{-1} \cdot Z_f(\varphi)(\sigma) = \frac{4}{pq} \cdot \varphi(0,0).$$

(iii) If $0 < p < 1 - a/b$, then there exists a constant $C(\varphi)$ which depends on f, φ but is independent of σ such that

$$\lim_{\sigma \to -1/b+0} Z_f(\varphi)(\sigma) = C(\varphi).$$

Here $C(\varphi)$ is positive if $\varphi(0,0) > 0$ and $\varphi(x,y) \geq 0$.

Corollary

Let f be as in (4) *and φ satisfy the condition: $\varphi(0,0) > 0$ and $\varphi(x,y) \geq 0$. Then $Z_f(\varphi)(s)$ cannot be meromorphically continued to any open neighborhood of $s = -1/b$ in $\mathbb{C}$. In particular, $m_0(f) = 1/b$ holds.*

Remark

The above theorem gives the optimality: $m_0(f) \geq 1/b$ in the case (D) with the condition: $2 \leq a < b$ in (D).

In the case when $a = b$, we have the following.

Theorem 2 (N.)

(i) If f belongs to the cases (B) or (C) with the condition: $a = b$, then the estimate: $m_0(f) \geq 1/a$ is optimal.

(ii) If f belongs to the case (D) with the condition: $a = b$ and $a \neq 2$, then the estimate: $m_0(f) \geq 1/a$ is optimal.

Key to proof of Theorem 2 (i)

Let us consider the following smooth functions belonging to the case (B).

$$f(x,y) = x^a y^a + x^a y^{a-q} e^{-1/|x|^p}, \tag{5}$$

where a, p, q satisfy that

- a is a positive integer satisfying $a \geq 2$,
- p is a positive real number,
- q is an even integer satisfying $2 \leq q \leq a$.

Recall

When $a = b$, the cases (B) and (C) are equivalent under switching the x and y variables.

In the cases, $Z_f(\varphi)(s)$ become holomorphic functions on the half-plane $\mathrm{Re}(s) > -1/a$.

Theorem

Let f be as in (5). Assume that $q < a$. Then the following hold:

(i) *If p is not an even integer, then*

$$\lim_{\sigma \to -1/a+0} \frac{a\sigma + 1}{|\log(a\sigma + 1)|} \left(Z_f(\varphi)(\sigma) - \sum_{j=0}^{M} \frac{c_j}{(a\sigma + 1)^{2-(2j)/p}} \right) = 4\varphi(0,0) \left(2 - \frac{1}{p} \right),$$

where $M = \max\{j \in \mathbb{Z}_{\geq 0} : 2j < p\}$ and $c_j = \frac{4}{(2j)!} \frac{\partial^{2j} \varphi}{\partial y^{2j}}(0,0)$.

(ii) If p is an even integer, then

$$\lim_{\sigma\to -1/a+0}\frac{a\sigma+1}{|\log(a\sigma+1)|}\left(Z_f(\varphi)(\sigma)-\sum_{j=0}^{M}\frac{c_j}{(a\sigma+1)^{2-(2j)/p}}\right)$$
$$=4\varphi(0,0)\left(2-\frac{1}{p}\right)+\frac{4}{p!q}\frac{\partial^p\varphi}{\partial y^p}(0,0).$$

where M, c_j are as above.

Corollary

Let f be as in (5). Assume that $q < a$. We suppose that one of the following hold:

(a) *p is not an even integer, $\varphi(0,0) \neq 0$ and $p \neq 1/2$;*

(b) *p is an even integer and*

$$\varphi(0,0)\left(2 - \frac{1}{p}\right) + \frac{1}{p!q}\frac{\partial^p \varphi}{\partial y^p}(0,0) \neq 0;$$

(c) *$p > 2$ and there is a nonnegative integer $j \in \{1, \ldots, M\}$ satisfying that $\frac{\partial^{2j}\varphi}{\partial y^{2j}}(0,0) \neq 0$.*

Then $Z_f(\varphi)(s)$ cannot be meromorphically continued to any open neighborhood of $s = -1/a$ in $\mathbb{C}$. In particular, $m_0(f) = 1/a$ holds.

Remark

The above corollary gives the optimality: $m_0(f) \geq 1/a$ in the case (B) (and (C)) with the condition: $a = b \geq 3$.

Theorem

Let f be as in (5). Assume that $q = a$ and $p > 2$. Then

$$\lim_{\sigma \to -1/a+0} (a\sigma + 1)^{2-(2M)/p} \left(Z_f(\varphi)(\sigma) - \sum_{j=0}^{M-1} \frac{c_j}{(a\sigma+1)^{2-(2j)/p}} \right) = c_M,$$

where $M = \max\{j \in \mathbb{Z}_{\geq 0} : 2j < p\}$ and $c_j = \frac{4}{(2j)!} \frac{\partial^{2j}\varphi}{\partial y^{2j}}(0,0)$.
In particular, $m_0(f) = 1/a$ if there is a nonnegative integer $j \in \{1, \dots, M\}$ satisfying that $\frac{\partial^{2j}\varphi}{\partial y^{2j}}(0,0) \neq 0$.

Remark

The above theorem gives the optimality: $m_0(f) \geq 1/a$ in the case (B) (and (C)) with the condition: $a = b = 2$.

Key to proof of Theorem 2 (ii)

Let us consider the following smooth functions belonging to (D).

$$f(x,y) = x^a y^a + x^a y^{a-q} e^{-1/|x|^p} + x^{a-\tilde{q}} y^a e^{-1/|y|^{\tilde{p}}}, \tag{6}$$

where $a, p, \tilde{p}, q, \tilde{q}$ satisfy that

- a is a positive integers satisfying $a \geq 2$,
- $p, \tilde{p}$ are positive real numbers,
- $q, \tilde{q}$ are even integers satisfying $2 \leq q \leq a$, $2 \leq \tilde{q} \leq a$.

Theorem

Let f be as in (6). Assume that $q < a$ and $\tilde{q} < a$. Then

$$\lim_{\sigma \to -1/a} \frac{a\sigma + 1}{|\log(a\sigma + 1)|} \left(Z_f(\varphi)(\sigma) - \frac{4\varphi(0,0)}{(a\sigma+1)^2} \right) = -4\varphi(0,0) \left(\frac{1}{p} + \frac{1}{\tilde{p}} \right).$$

Corollary

Let f be as in (6). Assume that $q < a$ and $\tilde{q} < a$. If $\varphi(0,0) \neq 0$, then $Z_f(\varphi)(s)$ cannot be meromorphically continued to any open neighborhood of $s = -1/a$ in $\mathbb{C}$. In particular, $m_0(f) = 1/a$ holds.

Remark

The above corollary gives the optimality: $m_0(f) \geq 1/a$ in the case (D) with the condition: $a = b \geq 3$.

The case (D) with the condition: $a = b = 2$ remains.

Question

Does the following hold ?: If f belongs to the case (D) with the condition $a = b = 2$, then the estimate: $m_0(f) \geq 1/2 \ (= 1/a)$ is optimal.

It is naturally expected that the value of $m_0(f)$ is determined by the optimal lower bound in each case.

Question

Do the following equalities hold ?

(i) For all f belonging to the case (B) or (D), $m_0(f) = 1/b$ with $b > 0$;

(ii) For all f belonging to the case (C), $m_0(f) = 1/a$ with $a > 0$.

Thank you for your attention!